University of California, Berkeley
Physics H7B Spring 1999 (*Strovink*)

SOLUTION TO PROBLEM SET 2

1. RHK problem 24.18
Solution: For ease of notation, here we denote the mean of any function $f(v)$ of the speed v of a gas molecule by

$$\langle f(v)\rangle \equiv \frac{\int_0^\infty f(v')n(v')dv'}{\int_0^\infty n(v')dv'}$$

where $n(v)$ (called dN/dv in lecture) is the distribution of v. If this formula is used, $n(v)$ *does not need to be normalized*. With this notation, for example, $\bar{v} \equiv \langle v\rangle$. Proceeding with the problem,

$$\begin{aligned}
v_{\rm rms} &\equiv \sqrt{\langle v^2\rangle} \quad \text{(RHK Eq. 23.15)}\\
0 &\le \langle (v-\bar{v})^2\rangle\\
\langle (v-\bar{v})^2\rangle &= \langle v^2\rangle - \langle 2v\bar{v}\rangle + \langle \bar{v}^2\rangle\\
&= \langle v^2\rangle - \bar{v}\langle 2v\rangle + \bar{v}^2\\
&= \langle v^2\rangle - 2\bar{v}^2 + \bar{v}^2\\
&= \langle v^2\rangle - \bar{v}^2\\
0 &\le \langle v^2\rangle - \bar{v}^2\\
\bar{v}^2 &\le \langle v^2\rangle\\
\bar{v} &\le \sqrt{\langle v^2\rangle}\\
\bar{v} &\le v_{\rm rms}\ .
\end{aligned}$$

The equality occurs only when $\langle (v-\bar{v})^2\rangle = 0$, *i.e.* all the molecules have the average speed $\bar{v}$.

2. RHK problem 24.21
Solution: Using the notation introduced above,
(b)

$$\begin{aligned}
\langle v\rangle &= \frac{\int_0^{v_0} v'Cv'^2dv'}{\int_0^{v_0} Cv'^2dv'}\\
&= \frac{\frac{1}{4}v_0^4}{\frac{1}{3}v_0^3}\\
&= \frac{3}{4}v_0\ .
\end{aligned}$$

(c)

$$\begin{aligned}
\langle v^2\rangle &= \frac{\int_0^{v_0} v'^2Cv'^2dv'}{\int_0^{v_0} Cv'^2dv'}\\
&= \frac{\frac{1}{5}v_0^5}{\frac{1}{3}v_0^3}\\
&= \frac{3}{5}v_0^2\\
v_{\rm rms} &\equiv \sqrt{\langle v^2\rangle}\\
v_{\rm rms} &= \sqrt{\frac{3}{5}}v_0\ .
\end{aligned}$$

(a)

$$\begin{aligned}
N &\equiv \int_0^{v_0} Cv'^2dv'\\
&= \frac{1}{3}Cv_0^3\\
\frac{3N}{v_0^3} &= C\ .
\end{aligned}$$

3. RHK problem 24.25
Solution:

$$\begin{aligned}
n(E) &\propto E^{1/2}\exp{(-E/kT)} \quad \text{(RHK Eq. 24.27)}\\
E_{\rm rms} &\equiv \sqrt{\langle E^2\rangle}\\
\langle E^2\rangle &= \frac{\int_0^\infty E'^2E'^{1/2}\exp{(-E'/kT)}dE'}{\int_0^\infty E'^{1/2}\exp{(-E'/kT)}dE'}\\
\beta &\equiv 1/kT\\
\langle E^2\rangle &= \frac{\int_0^\infty E'^{5/2}\exp{(-\beta E')}dE'}{\int_0^\infty E'^{1/2}\exp{(-\beta E')}dE'}\\
&= \frac{\left(d^2/d\beta^2\right)\left(\int_0^\infty E'^{1/2}\exp{(-\beta E')}dE'\right)}{\int_0^\infty E'^{1/2}\exp{(-\beta E')}dE'}\\
Z &\equiv \int_0^\infty E'^{1/2}\exp{(-\beta E')}dE'\\
\langle E^2\rangle &= \frac{d^2Z/d\beta^2}{Z}\ .
\end{aligned}$$

The remaining definite integral Z has dimension (energy)$^{3/2}$. Since the limits of the integral are not finite, the only available quantity with which a dimensionful scale may be set is β, which has dimension 1/energy. Therefore the integral must be equal to $\beta^{-3/2}$ multiplied by some constant C:

$$\begin{aligned}
\langle E^2\rangle &= \frac{\left(d^2/d\beta^2\right)\left(C\beta^{-3/2}\right)}{C\beta^{-3/2}} \\
&= \frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(C\beta^{-7/2}\right)}{C\beta^{-3/2}} \\
&= \frac{15}{4}\beta^{-2} \\
E_{\rm rms} &= \sqrt{\frac{15}{4}}\beta^{-1} \\
&= \sqrt{\frac{15}{4}}kT\ .
\end{aligned}$$

4. RHK problem 23.17
Solution: In Physics H7B, all problems involving numbers should be solved completely in terms of algebraic symbols before any numbers are plugged in (otherwise it is much more difficult to give part credit). Let
T = temperature of interstellar space = 2.7 °K
M = molar mass of H_2 = 0.0020 kg/mole (RHK Table 23.1)
$N_{\rm A}$ = Avogadro constant
= 6.022×10^{23} molecules/mole
m = mass of H_2 molecule = $M/N_{\rm A}$
$k_{\rm B}$ = Boltzmann constant = 1.38×10^{-23} J/K
Then from RHK Eq. 23.20,

$$\begin{aligned}
\frac{1}{2}m\langle v^2\rangle &= \frac{3}{2}k_{\rm B}T \\
\langle v^2\rangle &= \frac{3k_{\rm B}T}{m} \\
v_{\rm rms} &\equiv \sqrt{\langle v^2\rangle} \\
&= \sqrt{\frac{3k_{\rm B}T}{m}} \\
&= \sqrt{\frac{3k_{\rm B}N_{\rm A}T}{M}} \\
&= 183.5\ \text{m/sec}\ .
\end{aligned}$$

5. RHK problem 23.33
Solution: Let
$R_{\rm e}$ = radius of earth = 6.37×10^6 m
$R_{\rm m}$ = radius of moon = 1.74×10^6 m
$GM_{\rm e}/R_{\rm e}^2 = g$ = gravitational acceleration at earth's surface = 9.81 m/sec^2
$g_{\rm m}$ = gravitational acceleration at moon's surface = $0.16g$
$v_{\rm esc}$ = escape velocity at earth's surface
m = generic molecular mass
$N_{\rm A}$ = Avogadro constant
= 6.022×10^{23} molecules/mole
$M_{\rm Hyd}$ = molar mass of H_2 = 0.0020 kg/mole (RHK Table 23.1)
$M_{\rm Oxy}$ = molar mass of O_2 = 0.0320 kg/mole (RHK Table 23.1)
$k_{\rm B}$ = Boltzmann constant = 1.38×10^{-23} J/K
$T_{\rm esc}^{\rm Hyd}$(earth) = temperature (°K) at which rms H_2 velocity is equal to escape velocity at earth's surface
$T_{\rm esc}^{\rm Oxy}$(earth) = temperature ((°K) at which rms O_2 velocity is equal to escape velocity at earth's surface
$T_{\rm esc}^{\rm Hyd}$(m) = temperature ((°K) at which rms H_2 velocity is equal to escape velocity at moon's surface
$T_{\rm esc}^{\rm Oxy}$(m) = temperature ((°K) at which rms O_2 velocity is equal to escape velocity at moon's surface
Then

$$\begin{aligned}
\frac{1}{2}mv_{\rm esc}^2 &= \frac{GM_{\rm e}m}{R_{\rm e}} \\
v_{\rm esc}^2 &= \frac{2GM_{\rm e}}{R_{\rm e}} \\
&= 2gR_{\rm e} \\
\frac{1}{2}mv_{rms}^2 &= \frac{3}{2}k_{\rm B}T_{\rm esc} \\
v_{\rm rms} &= v_{\rm esc}\ \ \text{(stated by problem)} \\
\frac{1}{2}mv_{esc}^2 &= \frac{3}{2}k_{\rm B}T_{\rm esc} \\
\frac{1}{2}m\,2gR_{\rm e} &= \frac{3}{2}k_{\rm B}T_{\rm esc} \\
\frac{2mgR_{\rm e}}{3k_{\rm B}} &= T_{\rm esc}\ .
\end{aligned}$$

We use this general result to evaluate each of the four cases posed:

$$
\begin{aligned}
m &= \frac{M_{Hyd}}{N_{\mathrm{A}}} \\
T_{\mathrm{esc}}^{\mathrm{Hyd}}(\mathrm{earth}) &= \frac{2M_{\mathrm{Hyd}}gR_{\mathrm{e}}}{3k_{\mathrm{B}}N_{\mathrm{A}}} \\
&= 1.003 \times 10^4\ {}^\circ\mathrm{K} \\
T_{\mathrm{esc}}^{\mathrm{Oxy}}(\mathrm{earth}) &= \frac{2M_{\mathrm{Oxy}}gR_{\mathrm{e}}}{3k_{\mathrm{B}}N_{\mathrm{A}}} \\
&= 1.604 \times 10^5\ {}^\circ\mathrm{K} \\
T_{\mathrm{esc}}^{\mathrm{Hyd}}(\mathrm{moon}) &= \frac{2M_{\mathrm{Hyd}}g_{\mathrm{m}}R_{\mathrm{m}}}{3k_{\mathrm{B}}N_{\mathrm{A}}} \\
&= 438\ {}^\circ\mathrm{K} \\
T_{\mathrm{esc}}^{\mathrm{Oxy}}(\mathrm{moon}) &= \frac{2M_{\mathrm{Oxy}}g_{\mathrm{m}}R_{\mathrm{m}}}{3k_{\mathrm{B}}N_{\mathrm{A}}} \\
&= 7011\ {}^\circ\mathrm{K}\ .
\end{aligned}
$$

At an altitude in the Earth's atmosphere where the temperature is ≈ 1000 K, the preceding results imply that the rms velocity would be only a factor $\sqrt{T_{\mathrm{esc}}/T} \approx \sqrt{10}$ below the escape velocity; because of leakage out of the tail of the velocity distribution, little hydrogen would be expected to remain. For oxygen, the rms velocity would be a factor $\approx \sqrt{160}$ below the escape velocity, allowing that molecule to survive as an atmospheric component.

6. RHK problem 23.37
Solution: For path 1, the work W done on the gas is

$$
\begin{aligned}
W &= -\int_{\mathrm{path}} p\,dV \quad (\text{RHK } 23.24) \\
&= -\int_2^8 p\,dV - \int_8^8 p\,dV - \int_8^2 p\,dV \\
&= -(12.5\ \mathrm{kPa})(6\ \mathrm{m}^3) - 0 + (20\ \mathrm{kPa})(6\ \mathrm{m}^3) \\
&= 45\ \mathrm{kJ}\ ,
\end{aligned}
$$

where we have evaluated each straight-line segment by reading $\langle p \rangle$ off the graph, multiplying it by the difference in V to compute the area under the line. Similarly, for path 2,

$$
\begin{aligned}
W &= -\int_{\mathrm{path}} p\,dV \\
&= -\int_2^8 p\,dV - \int_8^2 p\,dV - \int_2^2 p\,dV \\
&= -(12.5\ \mathrm{kPa})(6\ \mathrm{m}^3) + (5\ \mathrm{kPa})(6\ \mathrm{m}^3) - 0 \\
&= -45\ \mathrm{kJ}\ .
\end{aligned}
$$

7. RHK problem 25.16
Solution: Let
m_v = (unknown) mass of vaporized material (ice), in kg
m_f = mass of fused material (ice) = 0.15 kg
L_v = latent heat of vaporization of water = 2256×10^3 J/kg
L_f = latent heat of fusion of water = 333×10^3 J/kg
c = specific heat capacity of water = 4190 J/kg·C°
T_v = temperature of steam = 100 °C
T_f = temperature of ice = 0 °C
T = final temperature of steam-ice mixture = 50 °C
The fact that the container is thermally insulated means that the total heat Q transferred out of the steam molecules is transferred into the ice molecules:

$$
\begin{aligned}
Q(\text{lost by steam}) &= Q(\text{gained by ice}) \\
m_v(L_v + c(T_v - T)) &= m_f(L_f + c(T - T_f)) \\
m_v &= m_f \frac{L_f + c(T - T_f)}{L_v + c(T_v - T)} \\
&= 0.033\ \mathrm{kg}\ .
\end{aligned}
$$

8. RHK problem 25.21
Solution: Let
Q = (unknown) heat transferred into sample
T_i = initial temperature = 6.6 K
T_f = final temperature = 15 K
m = mass of Al = 0.0012 kg
C = heat capacity per mole of Al
η = coefficient of T^3 in expression for C = 3.16×10^{-5} J/mole·K^4
M_{Al} = molar mass of Al = 0.0270 kg/mole (RHK Appendix D)
c = heat capacity per kg of Al = C/M_{Al}

With these definitions,

$$
\begin{aligned}
Q &= m \int_{T_i}^{T_f} c(T)\, dT \quad \text{(RHK Eq. 25.4)} \\
&= \frac{m}{M_{\text{Al}}} \int_{T_i}^{T_f} C(T)\, dT \\
C(T) &= \eta T^3 \\
Q &= \frac{m}{M_{\text{Al}}} \eta \int_{T_i}^{T_f} T^3 dT \\
&= \frac{m\eta}{4M_{\text{Al}}} \left(T_f^4 - T_i^4\right) \\
&= 0.0171 \text{ J} \,.
\end{aligned}
$$